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Simplified TBA equations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model

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Abstract: We use the recently found integral representation for the dressing phase in the kinematic region of the mirror theory to simplify the TBA equations for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model. The resulting set of equations provides an efficient starting point for both analytic and numerical studies.

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Contents

1 Introduction

Recently there has been a substantial progress towards solving the finite-size spectral problem of the AdS/CFT correspondence [1]. First, a perturbative approach due to Lüscher has been generalized to the case of the non-Lorentz invariant light-cone string sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and further applied to find the four- and five-loop anomalous dimensions of the Konishi operator [2, 3]; the four-loop result exhibits a remarkable agreement with the direct field-theoretic computation [4, 5]. Second, the TBA approach [6] based on the so-called $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model [7] has been advanced as a mean to determine the *exact* string spectrum.[1] In particular, the TBA equations for the ground state of the light-cone superstring were derived in [9]–[12]. Another important tool for studying the finite-size spectral problem, namely, the so-called Y-system has been proposed in [13], and its general solution has been constructed in [14]. Upon specifying an analytic behavior, solutions of the Y-system should also describe the excited states of the model. Comparison of the TBA equations to those of the Y-system [10] reveals intricate analytic properties of the latter indicating that, in contrast to relativistic models, the corresponding Y-system should be defined on an infinite genus Riemann surface [15]. Finally, we point out that in the work [12] integral (TBA-like) equations for excited states in the $\mathfrak{sl}(2)$ sector have been suggested along the lines of [2, 16–18] and they were further used in [19] to compute numerically an all-loop anomalous dimension supposedly corresponding to one of the descendents of the Konishi operator. The subleading at strong coupling $\lambda^{-1/4}$-term found from this computation disagrees however with the result by [20] and the origin of this disagreement remains unclear for the moment. Some important subtleties concerning the non-analytic behavior of the asymptotic string energies at strong coupling have been pointed out in the recent work [21].

[1] The importance of the TBA approach in the AdS/CFT spectral problem was stressed in [8] where it was used to explain wrapping effects in gauge theory.

Needless to say that the TBA/Y-system equations proposed above have a number of unusual features which call for a deeper understanding of their structure and analytic properties. In this note we will make a further step in simplifying the TBA equations which follow from the corresponding string hypothesis [9] by using the canonical procedure [22]. We recall that the Y-system is obtained from the *canonical* TBA equations by acting on the latter with the discrete Laplace operator Δ_{MN}, where $M, M = 1, \ldots, \infty$. The Laplace operator has the following representation $\Delta_{MN} = (K+1)^{-1}_{MN} \star s^{-1}$, where $K+1$ is a certain invertible operator and s^{-1} is an operator which has a null space, i.e. in general $f \star s^{-1} \star s \neq f$. The fist simplification of the canonical TBA system occurs when acting on it with $K+1$, because it brings most of the TBA equations to the local form, see [10] for details.

There are further simplifications we point out in this note. The first one concerns the infinite sums involving the Y-function $Y_{M|vw}$ and $Y_{M|w}$ for the so-called $M|vw$- and $M|w$-strings [10]. These infinite sums occur in some of the TBA equations and they are difficult for numerical studies due to their rather slow convergence properties. We show that by using certain identities between the TBA kernels these sums can be removed in favor of infinite sums involving Y_Q-functions only, the latter have much better convergence properties.

The second simplification concerns the main TBA equation for the Q-particles which involves the contribution of the dressing phase [23]. This phase is nothing else but the BES expression [24] analytically continued to the kinematic region of the mirror theory [7]. In [25] we have obtained a convenient integral representation for this analytic continuation starting from the DHM representation [26] valid in the kinematic region of the original string theory. Here we will work out explicitly the action of the operator $(K+1)^{-1}$ on the mirror dressing phase given by this integral representation and find a very simple final expression.

We believe that the simplification procedure developed here can also be applied to the integral equations describing the excited states, although there are new important subtleties related to singularities of certain Y-functions that should be taken into account.

The note is organized as follows. In the next section we present the main result on the simplified TBA equations. The interested reader can find the details of our derivation in two appendices.

2 Simplified TBA equations

We recall [10] that the spectrum of the mirror model in the thermodynamic limit contains Q-particles with pseudo-energy ϵ_Q, two copies of $M|w$- and $M|vw$-strings with pseudo-energies $\epsilon^{(\alpha)}_{M|w}$ and $\epsilon^{(\alpha)}_{M|vw}$, where $\alpha = 1, 2$, and, finally, two copies of $y^{\pm}$-particles, whose pseudo-energies $\epsilon^{(\alpha)}_{y^{\pm}}$ are supported on the interval $[-2, 2]$ of the rapidity variable u. The pseudo-energies and densities for all the other particles are defined for all real values of u. It is convenient to introduce so-called Y-functions which are related to the pseudo-energies as

$$Y_Q = e^{-\epsilon_Q}\,, \quad Y^{(\alpha)}_{M|vw} = e^{\epsilon^{(\alpha)}_{M|vw}}\,, \quad Y^{(\alpha)}_{M|w} = e^{\epsilon^{(\alpha)}_{M|w}}\,, \quad Y^{(\alpha)}_{\pm} = e^{\epsilon^{(\alpha)}_{y^{\pm}}}\,, \quad \alpha = 1, 2\,. \tag{2.1}$$

By using the integral representation for the mirror model dressing factor [25], the partially simplified set of the TBA equations obtained in [10] can be brought to the form

- $M|w$-strings: $M \geq 1$, $Y^{(\alpha)}_{0|w} = 0$

$$\log Y^{(\alpha)}_{M|w} = \log(1 + Y^{(\alpha)}_{M-1|w})(1 + Y^{(\alpha)}_{M+1|w}) \star s + \delta_{M1} \log \frac{1 - \frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1 - \frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}} \star s \tag{2.2}$$

- $M|vw$-strings: $M \geq 1$, $Y^{(\alpha)}_{0|vw} = 0$

$$\begin{aligned}\log Y^{(\alpha)}_{M|vw} = {}& \log(1 + Y^{(\alpha)}_{M-1|vw})(1 + Y^{(\alpha)}_{M+1|vw}) \star s \\ & - \log(1 + Y_{M+1}) \star s + \delta_{M1} \log \frac{1 - e^{-ih_\alpha} Y^{(\alpha)}_-}{1 - e^{-ih_\alpha} Y^{(\alpha)}_+} \star s\end{aligned} \tag{2.3}$$

- y-particles

$$\log Y^{(\alpha)}_\pm = -\log\left(1 + Y_Q\right) \star K^{Qy}_\pm + \log \frac{1 + \frac{1}{Y^{(\alpha)}_{M|vw}}}{1 + \frac{1}{Y^{(\alpha)}_{M|w}}} \star K_M \tag{2.4}$$

- Q-particles for $Q \geq 2$

$$\log Y_Q = \log \frac{\left(1 + \frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1 + \frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{(1 + \frac{1}{Y_{Q-1}})(1 + \frac{1}{Y_{Q+1}})} \star s \tag{2.5}$$

- $Q = 1$-particle

$$\log Y_1 = \log \frac{\left(1 - \frac{e^{ih_1}}{Y^{(1)}_-}\right)\left(1 - \frac{e^{ih_2}}{Y^{(2)}_-}\right)}{1 + \frac{1}{Y_2}} \star s - \check{\Delta} \star s\,, \tag{2.6}$$

where

$$\begin{aligned}\check{\Delta} = {}& L\,\check{\mathcal{E}} + \log\left(1 - \frac{e^{ih_1}}{Y^{(1)}_-}\right)\left(1 - \frac{e^{ih_2}}{Y^{(2)}_-}\right)\left(1 - \frac{e^{ih_1}}{Y^{(1)}_+}\right)\left(1 - \frac{e^{ih_2}}{Y^{(2)}_+}\right) \star \check{K} \\ & + \log\left(1 + \frac{1}{Y^{(1)}_{M|vw}}\right)\left(1 + \frac{1}{Y^{(2)}_{M|vw}}\right) \star \check{K}_M + 2\log\left(1 + Y_Q\right) \star \check{K}^\Sigma_Q\,.\end{aligned} \tag{2.7}$$

Let us stress that in the convolutions involving $Y^{(\alpha)}_\pm$-functions one has to integrate over the interval $[-2, 2]$. In eq. (2.7) L coincides with the light-cone momentum P_+ of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory in the light-cone gauge, which is simultaneously the circumference of a cylinder on which the corresponding string sigma model is defined. In this paper we consider only the $a = 0$ light-cone gauge (or temporal gauge) [27, 28] where $L = J$, and J is one of the SO(6) charges carried by the string. Also, $h_\alpha = (-1)^\alpha h$, where h can be thought of as the chemical potential for fermionic particles.

JHEP11(2009)019

The energy of the ground state of the light-cone gauge-fixed string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ defined on a cylinder of circumference L is expressed through the Y-functions which solve the TBA equations as follows

$$E_h(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi}\frac{d\widetilde{p}_Q}{du}\log\left(1+Y_Q\right) . \tag{2.8}$$

Equations above involve convolutions with a number of kernels which we specify in appendix A.

The TBA equations (2.4) for $y_\pm$-particles contain the infinite sum involving the Y-functions for vw- and w-strings. It can be replaced by a sum of terms which only involve $Y^{(\alpha)}_{1|vw}$, $Y^{(\alpha)}_{1|w}$ and Y_Q by using the following formula derived in [10]

$$\begin{aligned}\log\frac{1+\frac{1}{Y^{(\alpha)}_{M|vw}}}{1+\frac{1}{Y^{(\alpha)}_{M|w}}}\star K_M &= \log\frac{1+Y^{(\alpha)}_{1|vw}}{1+Y^{(\alpha)}_{1|w}}\star s \\ &\quad+\log(1+Y_{Q+1})\star s\star K_Q+\log(1+Y_Q)\star K_{Qy}\star s\star K_1 .\end{aligned} \tag{2.9}$$

Since Y_Q-functions decrease very fast for large Q, the formula (2.9) seems to be useful for numerical studies of the TBA equations. Let us also mention that the last sum in eq. (2.9) can be expressed in terms of $Y_\pm$-functions by using the formula that follows from eq. (2.4)

$$\log(1+Y_Q)\star K_{Qy} = \log\frac{Y^{(\alpha)}_+}{Y^{(\alpha)}_-} = \frac{1}{2}\log\frac{Y^{(1)}_+}{Y^{(1)}_-}\frac{Y^{(2)}_+}{Y^{(2)}_-}\,, \qquad \alpha=1,2 . \tag{2.10}$$

The kernel $\check{K}^{\Sigma}_Q$ corresponding to the improved dressing factor is worked out in appendix B. It has the following representation

$$\check{K}^{\Sigma}_Q = -K_{Qy}\star\check{I}_0+\check{I}_Q\,,$$

where $\check{I}_Q$ is given by eq. (B.36). Due to this representation of $\check{K}^{\Sigma}_Q$, the formula (2.10) can be also used to partially exclude the infinite contribution of Q-particles in eq. (2.7) in favor of the y-particles.

Finally, $\check{\Delta}$ in the TBA equation (2.6) contains another infinite sum involving the Y-functions for vw-strings. This sum can be expressed in terms of Y_Q-functions only by using the following identity that holds outside the interval $[-2,2]$

$$\begin{aligned}\log\left(1+\frac{1}{Y^{(1)}_{M|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{M|vw}}\right)\star\check{K}_M &= \log\left(1+Y_Q\right)\left(1+Y_{Q+2}\right)\star\check{K}_Q \\ &\quad+\log Y_2+2\log Y_2\star\check{K}-\log Y_1\star\check{K}_1\,,\end{aligned} \tag{2.11}$$

where both sums in the left and the right hand side are from 1 to ∞. Note also that at large L the r.h.s. of (2.11) is finite because $\widetilde{\mathcal{E}}_2+2\widetilde{\mathcal{E}}_2\star\check{K}-\widetilde{\mathcal{E}}_1\star\check{K}_1=0$, and we recall that in the term $\widetilde{\mathcal{E}}_2\star\check{K}$ one integrates over the interval $[-2,2]$. In fact for the ground state the l.h.s. of (2.11) goes to 0 in the large L limit [15], and it can be also easily seen from the r.h.s. of (2.11) by using that $1\star\check{K}=-\frac{1}{2}\big(\theta(-u-2)+\theta(u+2)\big)$ and $1\star\check{K}_M=0$.

To prove this formula, one should use eq. (2.5), and the following identities which hold for $|v|>2$

$$s\star\big(\check{K}_{Q-1}+\check{K}_{Q+1}\big)=\check{K}_Q\,,\qquad Q=2,3,\cdots,\infty\,, \tag{2.12}$$

$$s\star\check{K}_2=\check{K}_1-s-2s\star\check{K}=\check{K}_1(u,v)-s(u-v)-2\int_{-2}^{2}dt\,s(u-t)\check{K}(t-v)\,. \tag{2.13}$$

As the result of these simplifications the quantity $\check{\Delta}$ in eq. (2.7) can be written in the form

$$\begin{aligned}\check{\Delta}\,=\,L\,\check{\mathcal{E}}+\log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(1-\frac{e^{ih_1}}{Y_+^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_+^{(2)}}\right)\star\check{K} \\ +\log Y_2+2\log Y_2\star\check{K}-\log Y_1\star\check{K}_1+\log\left(1+Y_Q\right)\star\big(2\check{K}_Q^{\Sigma}+\check{K}_Q+\check{K}_{Q-2}\big)\,.\end{aligned} \tag{2.14}$$

To recall, in the last formula the sums over Q run from 1 to ∞, the convolutions involving $\check{K}$ are taken over the interval $[-2,2]$, and we use the convention $\check{K}_{-1}=\check{K}_0=0$.

Thus, eqs. (2.9) and (2.11) allow one to exclude from all the TBA equations the infinite sums involving the functions $Y_{M|vw}^{(\alpha)}$ and $Y_{M|w}^{(\alpha)}$. The resulting set of equations is well-suited for both analytic and numerical studies. In particular, one could analyze the behavior of the energy (2.8) as a function of the complexified length L or coupling constant g.

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A Kernels

All kernels and S-matrices we are using are expressed in terms of the function $x(u)$

$$x(u)=\frac{1}{2}\left(u-i\sqrt{4-u^2}\right),\qquad \mathrm{Im}\,x(u)<0\,, \tag{A.1}$$

which maps the u-plane with the cuts $[-\infty,-2]\cup[2,\infty]$ onto the physical region of the mirror theory, and the function $x_s(u)$

$$x_s(u)=\frac{u}{2}\left(1+\sqrt{1-\frac{4}{u^2}}\right)\,,\quad |x_s(u)|\geq 1\,, \tag{A.2}$$

which maps the u-plane with the cut $[-2,2]$ onto the physical region of the string theory.

The momentum $\tilde{p}^Q$ and the energy $\tilde{\mathcal{E}}_Q$ of a mirror Q-particle are expressed in terms of $x(u)$ as follows

$$\widetilde{p}_Q = gx\left(u - \frac{i}{g}Q\right) - gx\left(u + \frac{i}{g}Q\right) + iQ\,, \qquad \tilde{\mathcal{E}}_Q = \log\frac{x(u-\frac{i}{g}Q)}{x(u+\frac{i}{g}Q)}\,. \tag{A.3}$$

The TBA equations discussed in section 2 involve convolutions with a number of kernels which we specify below, see also [10] for more details and the definition of the convolutions. First, all the TBA equations contain the following universal kernels

$$s(u) = \frac{1}{2\pi i}\frac{d}{du}\log S(u) = \frac{g}{4\cosh\frac{\pi g u}{2}}\,, \qquad S(u) = \tanh[\frac{\pi}{4}(ug - i)]\,, \tag{A.4}$$

$$K_Q(u) = \frac{1}{2\pi i}\frac{d}{du}\log S_Q(u) = \frac{1}{\pi}\frac{g\,Q}{Q^2 + g^2u^2}\,, \qquad S_Q(u) = \frac{u - \frac{iQ}{g}}{u + \frac{iQ}{g}}\,, \tag{A.5}$$

which appear in TBA equations of any integrable model. Note that the kernel K_Q has an interesting group property

$$K_Q \star K_{Q'} = K_{Q'} \star K_Q = K_{Q+Q'}\,,$$

where the integrals in the convolution are taken from $-\infty$ to $+\infty$.

Then, the kernels $K_\pm^{Qy}$ are related to the scattering matrices $S_\pm^{Qy}$ of Q- and $y_\pm$-particles in the usual way

$$\begin{aligned} K_-^{Qy}(u,v) &= \frac{1}{2\pi i}\frac{d}{du}\log S_-^{Qy}(u,v)\,, \quad S_-^{Qy}(u,v) = \frac{x(u-i\frac{Q}{g}) - x(v)}{x(u+i\frac{Q}{g}) - x(v)}\sqrt{\frac{x(u+i\frac{Q}{g})}{x(u-i\frac{Q}{g})}}\,,\\ K_+^{Qy}(u,v) &= \frac{1}{2\pi i}\frac{d}{du}\log S_+^{Qy}(u,v)\,, \quad S_+^{Qy}(u,v) = \frac{x(u-i\frac{Q}{g}) - \frac{1}{x(v)}}{x(u+i\frac{Q}{g}) - \frac{1}{x(v)}}\sqrt{\frac{x(u+i\frac{Q}{g})}{x(u-i\frac{Q}{g})}}\,. \end{aligned} \tag{A.6}$$

These kernels can be expressed in terms of the kernel K_Q, and the kernel

$$K(u,v) = \frac{1}{2\pi i}\frac{d}{du}\log\left(\frac{x(u) - x(v)}{x(u) - 1/x(v)}\right) = \frac{1}{2\pi i}\frac{\sqrt{4-v^2}}{\sqrt{4-u^2}}\frac{1}{u-v}\,, \tag{A.7}$$

as follows

$$K_\mp^{Qy}(u,v) = \frac{1}{2}\Big(K_Q(u-v) \pm K_{Qy}(u,v)\Big)\,, \tag{A.8}$$

where K_{Qy} is given by

$$K_{Qy}(u,v) = K\left(u - \frac{i}{g}Q, v\right) - K\left(u + \frac{i}{g}Q, v\right)\,. \tag{A.9}$$

Next, we introduce the following kernel

$$\bar{K}(u,v) = \frac{1}{2\pi i}\frac{d}{du}\log\left(\frac{x(u) - x_s(v)}{x(u) - 1/x_s(v)}\right) = \frac{1}{2\pi}\frac{\sqrt{1-\frac{4}{v^2}}}{\sqrt{4-u^2}}\frac{v}{u-v}\,, \tag{A.10}$$

This kernel (A.10) can be thought of as an analytic continuation of $K(u,v)$ from the mirror theory v-plane to the string theory one. With the help of this kernel we can now define[2]

$$\check{K}(u,v) = \bar{K}(u,v)\big[\theta(-v-2)+\theta(v-2)\big]\,, \tag{A.11}$$

$$\check{K}_Q(u,v) = \Big[\bar{K}\Big(u+\frac{i}{g}Q,v\Big)+\bar{K}\Big(u-\frac{i}{g}Q,v\Big)\Big]\big[\theta(-v-2)+\theta(v-2)\big]\,, \tag{A.12}$$

where $\theta(u)$ is the standard unit step function. Obviously, both $\check{K}$ and $\check{K}_Q$ vanish for v being in the interval $(-2,2)$ and are equal to (twice) the jump discontinuity of the kernels K and K_{Qy} across the real semi-lines $|v|>2$.

The quantity $\check{\mathcal{E}}$ is defined as

$$\check{\mathcal{E}}(u) = \log\frac{x(u-i0)}{x(u+i0)} = 2\log|x_s(u)| \neq 0 \quad \text{for} \quad |u|>2\,. \tag{A.13}$$

Finally, eq. (2.6) involves the kernel

$$\check{K}^{\Sigma}_Q = \frac{1}{2\pi i}\frac{\partial}{\partial u}\log\check{\Sigma}_Q = -K_{Qy}\star\check{I}_0+\check{I}_Q \tag{A.14}$$

where

$$\check{I}_Q = \sum_{n=1}^{\infty}\check{K}_{2n+Q}(u,v) = K_\Gamma^{[Q+2]}(u-v)+2\int_{-2}^{2}\mathrm{d}t\,K_\Gamma^{[Q+2]}(u-t)\check{K}(t,v)\,, \tag{A.15}$$

$$K_\Gamma^{[Q]}(u) = \frac{1}{2\pi i}\frac{d}{du}\log\frac{\Gamma\big[\frac{Q}{2}-\frac{i}{2}gu\big]}{\Gamma\big[\frac{Q}{2}+\frac{i}{2}gu\big]} = \frac{g\gamma}{2\pi}+\sum_{n=1}^{\infty}\Big(K_{2n+Q-2}(u)-\frac{g}{2\pi n}\Big)\,. \tag{A.16}$$

The kernel (A.14) is related to the dressing kernel

$$K^{\Sigma}_{QQ'}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log\Sigma_{QQ'}(u,u')\,, \tag{A.17}$$

where $\Sigma_{QQ'}(u,v)$ is the improved dressing factor [25] obtained by fusing the $\mathfrak{sl}(2)$ S-matrices for individual constituents of the bound states in the mirror theory. The relation is given by

$$\check{K}^{\Sigma}_Q(u,v) = K^{\Sigma}_{Q1}\Big(u,v+\frac{i}{g}-i0\Big)+K^{\Sigma}_{Q1}\Big(u,v-\frac{i}{g}+i0\Big)-K^{\Sigma}_{Q2}(u,v)\,, \tag{A.18}$$

and will be proven in the next section.

B Simplifying the dressing kernel contribution

B.1 Φ and Ψ functions

Below we present the functions Φ and Ψ used to represent the dressing phase in the kinematic region of the mirror theory

$$\Phi(x_1,x_2) = i\oint\frac{\mathrm{d}w_1}{2\pi i}\oint\frac{\mathrm{d}w_2}{2\pi i}\frac{1}{(w_1-x_1)(w_2-x_2)}I(w_1,w_2)\,, \tag{B.1}$$

$$\Psi(x_1,x_2) = i\oint\frac{\mathrm{d}w}{2\pi i}\frac{1}{w-x_2}I(x_1,w)\,, \tag{B.2}$$

[2]The definitions of the kernels $\check{K}$ and $\check{K}_Q$ differ by the sign from the ones used in [10].

where

$$I(w_1,w_2)=\log\frac{\Gamma\big[1+\frac{i}{2}g\big(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\big)\big]}{\Gamma\big[1-\frac{i}{2}g\big(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\big)\big]} \tag{B.3}$$

and the integrals are over the unit circles. Both functions are discontinuous through the unit circle. Considering $\Psi(x_1,x_2)$ as a function of the rapidity variable u through $x_1\equiv x(u)$, where $x(u)$ is given by eq. (A.1), it has an infinite number of cuts located at $u\pm\frac{2i}{g}n$, $-2\le u\le 2$ and $n=1,2,\infty$. Both Φ and Ψ are discontinuous when x_1 or x_2 for Φ and x_2 for Ψ crosses the unit circle. The corresponding jump discontinuities are given in [25].

B.2 Improved dressing factor

As we have shown in our recent work [25], the improved dressing factor in the kinematic region of the mirror theory does not depend on the internal structure of a bound state employed in the fusion procedure. Also, a convenient integral representation for this factor has been found in [25], namely

$$\begin{aligned}\frac{1}{i}\log\Sigma_{QQ'}(y_1,y_2)&=\Phi(y_1^+,y_2^+)-\Phi(y_1^+,y_2^-)-\Phi(y_1^-,y_2^+)+\Phi(y_1^-,y_2^-)\\&-\frac{1}{2}\left(\Psi(y_1^+,y_2^+)+\Psi(y_1^-,y_2^+)-\Psi(y_1^+,y_2^-)-\Psi(y_1^-,y_2^-)\right)\\&+\frac{1}{2}\left(\Psi(y_2^+,y_1^+)+\Psi(y_2^-,y_1^+)-\Psi(y_2^+,y_1^-)-\Psi(y_2^-,y_1^-)\right)\\&+\frac{1}{i}\log\frac{i^Q\,\Gamma\big[Q'-\frac{i}{2}g\big(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\big)\big]}{i^{Q'}\Gamma\big[Q+\frac{i}{2}g\big(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\big)\big]}\frac{1-\frac{1}{y_1^+y_2^-}}{1-\frac{1}{y_1^-y_2^+}}\sqrt{\frac{y_1^+y_2^-}{y_1^-y_2^+}}\,.\end{aligned} \tag{B.4}$$

Here $y_{1,2}^{\pm}$ are parameters of Q and Q'-particle bound states in the mirror theory. The bound state parameters read as

$$y_1^+=x\left(u+\frac{i}{g}Q\right),\qquad y_1^-=x\left(u-\frac{i}{g}Q\right), \tag{B.5}$$

$$y_2^+=x\left(u'+\frac{i}{g}Q'\right),\qquad y_2^-=x\left(u'-\frac{i}{g}Q'\right) \tag{B.6}$$

In the next subsection we will use this integral representation for the dressing kernel together with the properties of other kernels involved in the TBA equation to simplify the dressing kernel contribution to the TBA equation (2.6) which contains this kernel.

B.3 Computing $\check{K}^{\Sigma}_{Q}$

In [10] we have conjectured the following relation

$$K^{\Sigma}_{QQ'}\star(K+1)^{-1}_{Q'Q''}\overset{?}{=}\delta_{1Q''}\check{K}^{\Sigma}_{Q}\star s\,, \tag{B.7}$$

where the kernel $\check{K}^{\Sigma}_{Q}(u,v)$ is supposed to vanish for $|v|<2$. With an explicit expression (B.4) for the dressing kernel at hand, we can now verify this conjecture and find $\check{K}^{\Sigma}_{Q}$.

Denote by $\Delta_{Q'Q''}$ the discrete Laplace operator

$$\Delta_{Q'Q''} \equiv (K+1)^{-1}_{Q'Q''} \star s^{-1} = \delta_{Q'Q''} s^{-1} - (\delta_{Q'+1,Q''} + \delta_{Q'-1,Q''}) \,. \tag{B.8}$$

For $Q'' = 1$ this is not anymore the Laplace operator, but we continue to use the same notation, i.e.,

$$\Delta_{Q'1} \equiv (K+1)^{-1}_{Q'1} \star s^{-1} = \delta_{Q'1} s^{-1} - \delta_{Q'2} \,. \tag{B.9}$$

As was shown in [25], the improved dressing factor is a holomorphic function of its arguments in the intersection of the region $\{|y^+_{1,2}| < 1, |y^-_{1,2}| > 1\}$ with the mirror region $\operatorname{Im} y^\pm_i < 0$, which includes the real momentum line of the mirror theory. This immediately implies that

$$K^\Sigma_{QQ'} \star \Delta_{Q'Q''} = 0 \quad \text{for} \quad Q'' \neq 1. \tag{B.10}$$

Now we consider $\frac{1}{i}\log \Sigma_{QQ'} \star \Delta_{Q'1}$. We have to distinguished two cases, $|v| < 2$ and $|v| > 2$. We start with the first case.

Case I: $|v| < 2$.

The formula (B.4) contains four lines which contributions we will work out separately. The computation proceeds as follows

$$\Phi(y_1^+, y_2^+) \star \Delta_{Q'1} = \Phi\left[y_1^+, x\left(v + \frac{i}{g}Q'\right)\right] \star \Delta_{Q'1} = \tag{B.11}$$

$$\Phi\left[y_1^+, x\left(v + \frac{2i}{g} - i0\right)\right] + \Phi\left[y_1^+, x(v+i0)\right] - \Phi\left[y_1^+, x\left(v + \frac{2i}{g}\right)\right] = \Phi\left[y_1^+, x(v+i0)\right].$$

Thus, for the difference of two Φ-functions with the same first argument, we find

$$\left(\Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-)\right) \star \Delta_{Q'1} = \Phi\left[y_1^+, x(v+i0)\right] - \Phi\left[y_1^+, x(v-i0)\right].$$

According to the formula (A.1), $x(v)$ has the property that $|x(v+iy)| < 1$ and $|x(v-iy)| > 1$, where v is real and $\operatorname{Im} y > 0$. Thus, the expression above equals to the jump discontinuity of the Φ-function through the unit circle. It is given by

$$\left(\Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-)\right) \star \Delta_{Q'1} = -\Psi(x(v), y_1^+)\,. \tag{B.12}$$

Proceeding in the similar manner, we obtain the contribution of the first line in eq. (B.4):

$$\begin{aligned}\left(\Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-) - \Phi(y_1^-, y_2^+) + \Phi(y_1^-, y_2^-)\right) \star \Delta_{Q'1} &= \\ = \Psi(x(v), y_1^-) - \Psi(x(v), y_1^+)\,.&\end{aligned} \tag{B.13}$$

Contribution of the second line in eq. (B.4) is computed exactly in the same fashion as of the first one. One should use this time the formula of [10] for the jump discontinuity of the Ψ-function, when its second argument crosses the unit circle. As a net result, we find

$$\begin{aligned}-\frac{1}{2}\left(\Psi(y_1^+, y_2^+) + \Psi(y_1^-, y_2^+) - \Psi(y_1^+, y_2^-) - \Psi(y_1^-, y_2^-)\right) \star \Delta_{Q'1} &= \\ = -\frac{1}{2i}\log\frac{u - v + \frac{i}{g}Q}{u - v - \frac{i}{g}Q} - \frac{1}{i}\log i^Q \frac{\Gamma\left[\frac{Q}{2} - \frac{i}{2}g(u-v)\right]}{\Gamma\left[\frac{Q}{2} + \frac{i}{2}g(u-v)\right]}\,.&\end{aligned} \tag{B.14}$$

The contribution of the third line in eq. (B.4) is a bit more tricky to figure out. The point is that the action of the second term in the operator $\Delta_{Q'1}$ puts the second argument of the function Ψ precisely on the cut of the latter located at $v+\frac{2i}{g}$. Therefore, to proceed, we have to specify which value of Ψ we utilize — ether the one on to the upper edge of the cut $v+\frac{2i}{g}+i0$ or the one on the lower edge $v+\frac{2i}{g}-i0$. Of course, the prescription should be fixed in the universal manner for all the Ψ-functions appearing in the third line of eq. (B.4).

It turns out, quite remarkably, that the net result does not depend on which prescription is used. As soon as a choice is made, the action of $\Delta_{Q'1}$ reduces to evaluation of the jump discontinuity of Ψ through the cut, which has been already done in [25]. We spear the details of the computation presenting only the final result

$$\begin{aligned}
&\frac{1}{2}\left(\Psi(y_2^+,y_1^+)+\Psi(y_2^-,y_1^+)-\Psi(y_2^+,y_1^-)-\Psi(y_2^-,y_1^-)\right)\star\Delta_{Q'1}= \\
&\qquad=\Psi(x(v),y_1^+)-\Psi(x(v),y_1^-)+\frac{1}{2i}\log\frac{y_1^+-x(v)}{y_1^+-\frac{1}{x(v)}}\frac{y_1^--\frac{1}{x(v)}}{y_1^--x(v)}\,.
\end{aligned}\tag{B.15}$$

The last term here can be also represented in the following form

$$\frac{1}{2i}\log\frac{y_1^+-x(v)}{y_1^+-\frac{1}{x(v)}}\frac{y_1^--\frac{1}{x(v)}}{y_1^--x(v)}=-\frac{1}{i}\log\frac{y_1^+-\frac{1}{x(v)}}{y_1^--\frac{1}{x(v)}}\sqrt{\frac{y_1^-}{y_1^+}}+\frac{1}{2i}\log\frac{u-v+\frac{i}{g}Q}{u-v-\frac{i}{g}Q}\,. \tag{B.16}$$

Finally, contribution of the forth line in eq. (B.4) is straightforward to find

$$\begin{aligned}
&\frac{1}{i}\log\frac{i^Q\,\Gamma\big[Q'-\frac{i}{2}g\big(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\big)\big]}{i^{Q'}\Gamma\big[Q+\frac{i}{2}g\big(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\big)\big]}\frac{1-\frac{1}{y_1^+y_2^-}}{1-\frac{1}{y_1^-y_2^+}}\sqrt{\frac{y_1^+y_2^-}{y_1^-y_2^+}}\star\Delta_{Q'1}= \\
&\qquad=\frac{1}{i}\log i^Q\frac{\Gamma\big[\frac{Q}{2}-\frac{i}{2}g(u-v)\big]}{\Gamma\big[\frac{Q}{2}+\frac{i}{2}g(u-v)\big]}+\frac{1}{i}\log\frac{y_1^+-\frac{1}{x(v)}}{y_1^--\frac{1}{x(v)}}\sqrt{\frac{y_1^-}{y_1^+}}\,.
\end{aligned}\tag{B.17}$$

Summing up all the contributions, we find zero, i.e.

$$\log\check{\Sigma}_Q(u,v)\equiv\log\Sigma_{QQ'}(y_1,y_2)\star\Delta_{Q'1}=0\quad\text{ for }\quad v\in(-2,2)\,.$$

Now we turn to the second case.

Case II: $|v|>2$.

In what follows we introduce the concise notation $x\equiv x(v-i0)$ which represents the (real) value of x on the lower edge of the cut $]-\infty,-2]\cup[2,\infty[$. The value on the upper edge is then $x(v+i0)=1/x(v-i0)$. Further, the function x can be conveniently represented as

$$x=\frac{1}{2}\big(v+\sqrt{v^2-4}\big)\theta(v-2)+\frac{1}{2}\big(v-\sqrt{v^2-4}\big)\theta(-v-2)\,.$$

Evaluation of the action of $\Delta_{Q'1}$ on $\log\Sigma_{QQ'}(y_1,y_2)$ does not contain any subtlety and we quote here the corresponding result

$$\begin{aligned}\frac{1}{i}\log\check{\Sigma}_Q(u,v) &= \frac{1}{i}\log\Sigma_{QQ'}(y_1,y_2)\star\Delta_{Q'1} = \\ &= \Phi(y_1^-,x) - \Phi\Big(y_1^-,\frac{1}{x}\Big) - \Phi(y_1^+,x) + \Phi\Big(y_1^+,\frac{1}{x}\Big) \\ &\quad + \frac{1}{2}\bigg(\Psi(y_1^-,x) - \Psi\Big(y_1^-,\frac{1}{x}\Big) + \Psi(y_1^+,x) - \Psi\Big(y_1^+,\frac{1}{x}\Big)\bigg) \\ &\quad + \Psi(x,y_1^+) - \Psi(x,y_1^-) \\ &\quad + \frac{1}{i}\log i^Q\frac{\Gamma\big[\frac{Q}{2}-\frac{i}{2}g(u-v)\big]}{\Gamma\big[\frac{Q}{2}+\frac{i}{2}g(u-v)\big]} + \frac{1}{i}\log\frac{y_1^+-\frac{1}{x}}{y_1^--x}\sqrt{x^2\frac{y_1^-}{y_1^+}}\,.\end{aligned} \tag{B.18}$$

Applying the derivative $\frac{1}{2\pi}\frac{d}{du}$ to the last formula, yields the kernel $\check{K}_Q^{\Sigma}$ for $|v|>2$.

It appears that the dressing kernel $\check{K}_Q^{\Sigma}=\frac{1}{2\pi i}\frac{d}{du}\log\check{\Sigma}_Q$ has a nice representation in terms of simpler kernels appearing in the TBA equations. To find it, we first note that since all the kernels, except for the dressing one, are defined on the (part of) real u-line, it is natural to transform the integration contours (circles) in the integrals entering eq. (B.18) into the interval $[-2,2]$. This is easily done by noting that for an arbitrary function $f(w)$ on a circle, $w=e^{i\theta}$, such that $f(w)=f(1/w)$ one has

$$\int\frac{dw}{2\pi i}\frac{1}{w-x}f(w) = \int_{-2}^{2}\frac{dz}{2\pi}\frac{1}{\sqrt{4-z^2}}\frac{2-zx}{x\big[x+\frac{1}{x}-z\big]}f(z)\,. \tag{B.19}$$

This formula, in conjunction with the identity

$$(1-x(v)^2)/x(v) = -\sqrt{v^2-4}\big[\theta(v-2)-\theta(-v-2)\big]$$

and the following properties of K_{Qy}

$$K_{Qy}(u,2) = K_{Qy}(u,-2) = 0\,, \tag{B.20}$$

allows one to derive the following formula

$$\begin{aligned}&\frac{\partial}{\partial u}\bigg[\Phi(y_1^-,x) - \Phi\Big(y_1^-,\frac{1}{x}\Big) - \Phi(y_1^+,x) + \Phi\Big(y_1^+,\frac{1}{x}\Big)\bigg] = \\ &\quad = \pm\frac{i}{\pi}\int_{-2}^{2}dt_1\frac{dt_2}{\sqrt{4-t_2^2}}K_{Qy}(u,t_1)\frac{\sqrt{v^2-4}}{t_2-v}\frac{d}{dt_1}\log\frac{\Gamma\big[1-\frac{i}{2}g(t_1-t_2)\big]}{\Gamma\big[1+\frac{i}{2}g(t_1-t_2)\big]}\,,\end{aligned} \tag{B.21}$$

where an overall " + " sign is for $v>2$ and " − " for $v<-2$, respectively. With the help of the kernel (A.16) the formula (B.21) can be written as the double convolution

$$\frac{1}{2\pi}\frac{\partial}{\partial u}\bigg[\Phi(y_1^-,x) - \Phi\Big(y_1^-,\frac{1}{x}\Big) - \Phi(y_1^+,x) + \Phi\Big(y_1^-,\frac{1}{x}\Big)\bigg] = -2K_{Qy}\star K_\Gamma^{[2]}\star\check{K}\,, \tag{B.22}$$

where both integrations are taken over the interval $[-2,2]$.

Now we investigate the second line of eq. (B.18). Performing the same steps as above, we find the following identity

$$\begin{aligned}\Delta\Psi &\equiv \frac{1}{2}\Big(\Psi(y_1^-,x)-\Psi\Big(y_1^-,\frac{1}{x}\Big)+\Psi(y_1^+,x)-\Psi\Big(y_1^+,\frac{1}{x}\Big)\Big) \\ &= \pm\frac{1}{2\pi i}\int_{-2}^{2}\frac{dt}{\sqrt{4-t^2}}\frac{\sqrt{v^2-4}}{t-v}\log\frac{\Gamma\big[1+\frac{Q}{2}-\frac{i}{2}g(u-t)\big]}{\Gamma\big[1+\frac{Q}{2}+\frac{i}{2}g(u-t)\big]}\frac{\Gamma\big[1-\frac{Q}{2}-\frac{i}{2}g(u-t)\big]}{\Gamma\big[1-\frac{Q}{2}+\frac{i}{2}g(u-t)\big]}\,,\end{aligned} \tag{B.23}$$

where again an overall " + " sign is for $v>2$ and " − " for $v<-2$, respectively. One further finds that

$$\frac{1}{2\pi i}\frac{d}{du}\log\frac{\Gamma\big[1+\frac{Q}{2}-\frac{i}{2}gu\big]}{\Gamma\big[1+\frac{Q}{2}+\frac{i}{2}gu\big]}\frac{\Gamma\big[1-\frac{Q}{2}-\frac{i}{2}gu\big]}{\Gamma\big[1-\frac{Q}{2}+\frac{i}{2}gu\big]}=K_\Gamma^{[Q+2]}+K_\Gamma^{[Q]}\,. \tag{B.24}$$

Furthermore,[3]

$$K_\Gamma^{[Q+2]}=K_\Gamma^{[Q]}-K_Q \qquad . \tag{B.25}$$

Therefore, the contribution corresponding to the second line takes the form

$$\frac{1}{2\pi}\frac{\partial}{\partial u}\Delta\Psi=(2K_\Gamma^{[Q]}-K_Q)\star\check{K}\,. \tag{B.26}$$

For the third line in eq. (B.18) we find

$$\frac{\partial}{\partial u}\Big[\Psi(x,y_1^+)-\Psi(x,y_1^-)\Big]=\mp i\int_{-2}^{2}dt\,\frac{\partial}{\partial t}K_{Qy}(u,t)\log\frac{\Gamma\big[1+\frac{i}{2}g(v-t)\big]}{\Gamma\big[1-\frac{i}{2}g(v-t)\big]}\,. \tag{B.27}$$

where an overall " + " sign is for $v<-2$ and " − " for $v>2$, respectively. Integrating by parts, we get

$$\frac{1}{2\pi}\frac{\partial}{\partial u}\Big[\Psi(x,y_1^+)-\Psi(x,y_1^-)\Big]=-K_{Qy}\star K_\Gamma^{[2]}\,, \tag{B.28}$$

Finally, we notice that for $|v|>2$ the following identity is valid

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\log\frac{y_1^+-\frac{1}{x}}{y_1^--x}\sqrt{x^2\frac{y_1^-}{y_1^+}}=-\frac{1}{2}\check{K}_Q(u,v)-\frac{1}{2}K_Q(u-v)\,. \tag{B.29}$$

Thus, for the last line in eq. (B.18) one gets

$$\begin{aligned}&\frac{1}{2\pi i}\frac{\partial}{\partial u}\log\frac{\Gamma\big[\frac{Q}{2}-\frac{i}{2}g(u-v)\big]}{\Gamma\big[\frac{Q}{2}+\frac{i}{2}g(u-v)\big]}\frac{y_1^+-\frac{1}{x}}{y_1^--x}\sqrt{x^2\frac{y_1^-}{y_1^+}} \\ &\qquad\qquad =K_\Gamma^{[Q]}(u-v)-\frac{1}{2}\check{K}_Q(u,v)-\frac{1}{2}K_Q(u-v)\,.\end{aligned} \tag{B.30}$$

[3] Another interesting relation is $K_Q\star K_\Gamma^{[2]}=K_\Gamma^{[Q+2]}$, where integration is performed over the whole real line.

Combining everything together, we find

$$\check{K}_Q^{\Sigma} = \frac{1}{2\pi i}\frac{\partial}{\partial u}\Sigma_Q(u,v) = -2K_{Qy}\star K_\Gamma^{[2]}\star\check{K} + (2K_\Gamma^{[Q]} - K_Q)\star\check{K} - K_{Qy}\star K_\Gamma^{[2]} + K_\Gamma^{[Q]} - \frac{1}{2}\check{K}_Q - \frac{1}{2}K_Q\,. \tag{B.31}$$

We stress again that all the convolutions here are taken from -2 to 2. The kernels involved in the last formula satisfy a number of magic properties, which lead to a significant simplification of eq. (B.31). First, one has

$$K_Q\star\check{K} = \frac{1}{2}\check{K}_Q - \frac{1}{2}K_Q\,, \qquad 1\star\check{K} = -\frac{1}{2}\,. \tag{B.32}$$

These relations allow one to find

$$K_\Gamma^{[Q]}\star\check{K} = \frac{1}{2}\check{K}_Q - \frac{1}{2}K_\Gamma^{[Q]} + \frac{1}{2}\sum_{n=1}^{\infty}\check{K}_{2n+Q}\,. \tag{B.33}$$

Specifying the last expression for $Q=2$, one gets

$$K_\Gamma^{[2]}\star\check{K} = \frac{1}{2}\check{K}_2 - \frac{1}{2}K_\Gamma^{[2]} + \frac{1}{2}\sum_{n=1}^{\infty}\check{K}_{2n+2} = -\frac{1}{2}K_\Gamma^{[2]} + \frac{1}{2}\sum_{n=1}^{\infty}\check{K}_{2n}\,. \tag{B.34}$$

Applying the identities (B.32)–(B.34) in the first line of eq. (B.31), we find the following simple result

$$\check{K}_Q^{\Sigma}(u,v) = -K_{Qy}\star\sum_{n=1}^{\infty}\check{K}_{2n} + \sum_{n=1}^{\infty}\check{K}_{2n+Q}\,. \tag{B.35}$$

We note that for numerical computations, the fastest algorithm consists in replacing the infinite sums in the last formula by their integral representation

$$\check{I}_Q = \sum_{n=1}^{\infty}\check{K}_{2n+Q}(u,v) = K_\Gamma^{[Q+2]}(u-v) + 2\int_{-2}^{2}\mathrm{d}t\,K_\Gamma^{[Q+2]}(u-t)\check{K}(t,v)\,. \tag{B.36}$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].

[2] Z. Bajnok and R.A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys.* **B 807** (2009) 625 [arXiv:0807.0399] [SPIRES].

[3] Z. Bajnok, A. Hegedus, R.A. Janik and T. Lukowski, *Five loop Konishi from AdS/CFT*, arXiv:0906.4062 [SPIRES].

[4] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Wrapping at four loops in $N=4$ SYM*, *Phys. Lett.* **B 666** (2008) 100 [arXiv:0712.3522] [SPIRES].

[5] V.N. Velizhanin, *The Four-Loop Konishi in $N=4$ SYM*, arXiv:0808.3832 [SPIRES].

[6] A.B. Zamolodchikov, *Thermodynamic Bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models*, Nucl. Phys. **B 342** (1990) 695 [SPIRES].

[7] G. Arutyunov and S. Frolov, *On String S-matrix, Bound States and TBA*, JHEP **12** (2007) 024 [arXiv:0710.1568] [SPIRES].

[8] J. Ambjørn, R.A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, Nucl. Phys. **B 736** (2006) 288 [hep-th/0510171] [SPIRES].

[9] G. Arutyunov and S. Frolov, *String hypothesis for the $AdS_5 \times S^5$ mirror*, JHEP **03** (2009) 152 [arXiv:0901.1417] [SPIRES].

[10] G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ Mirror Model*, JHEP **05** (2009) 068 [arXiv:0903.0141] [SPIRES].

[11] D. Bombardelli, D. Fioravanti and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal*, J. Phys. **A 42** (2009) 375401 [arXiv:0902.3930] [SPIRES].

[12] N. Gromov, V. Kazakov, A. Kozak and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT II*, arXiv:0902.4458 [SPIRES].

[13] N. Gromov, V. Kazakov and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT*, arXiv:0901.3753 [SPIRES].

[14] A. Hegedus, *Discrete Hirota dynamics for AdS/CFT*, arXiv:0906.2546 [SPIRES].

[15] S. Frolov and R. Suzuki, *Temperature quantization from the TBA equations*, Phys. Lett. **B 679** (2009) 60 [arXiv:0906.0499] [SPIRES].

[16] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Quantum field theories in finite volume: Excited state energies*, Nucl. Phys. **B 489** (1997) 487 [hep-th/9607099] [SPIRES].

[17] P. Dorey and R. Tateo, *Excited states by analytic continuation of TBA equations*, Nucl. Phys. **B 482** (1996) 639 [hep-th/9607167] [SPIRES].

[18] J. Teschner, *On the spectrum of the Sinh-Gordon model in finite volume*, Nucl. Phys. **B 799** (2008) 403 [hep-th/0702214] [SPIRES].

[19] N. Gromov, V. Kazakov and P. Vieira, *Exact AdS/CFT spectrum: Konishi dimension at any coupling*, arXiv:0906.4240 [SPIRES].

[20] R. Roiban and A.A. Tseytlin, *Quantum strings in $AdS_5 x S^5$: strong-coupling corrections to dimension of Konishi operator*, arXiv:0906.4294 [SPIRES].

[21] A. Rej and F. Spill, *Konishi at strong coupling from ABE*, J. Phys. **A 42** (2009) 442003 [arXiv:0907.1919] [SPIRES].

[22] F.H.L. Essler, H. Frahm, F. Göhmann, A. Klümper and V.E. Korepin, *The one-dimensional Hubbard model*, Cambridge University Press, Cambridge U.K. (2005).

[23] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, JHEP **10** (2004) 016 [hep-th/0406256] [SPIRES].

[24] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, J. Stat. Mech. (2007) P01021 [hep-th/0610251] [SPIRES].

[25] G. Arutyunov and S. Frolov, *The Dressing Factor and Crossing Equations*, J. Phys. **A 42** (2009) 425401 [arXiv:0904.4575] [SPIRES].

JHEP11(2009)019

[26] N. Dorey, D.M. Hofman and J.M. Maldacena, *On the singularities of the magnon S-matrix*, *Phys. Rev.* **D 76** (2007) 025011 [hep-th/0703104] [SPIRES].

[27] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$*, *JHEP* **02** (2005) 059 [hep-th/0411089] [SPIRES].

[28] G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 x S^5$ Superstring. Part I*, *J. Phys.* **A 42** (2009) 254003 [arXiv:0901.4937] [SPIRES].